Exam 2 Review (7.2-7.7)

Solve the following trigonometric integrals:

1. $\int sin^2(x)cos^3(x)\ dx$

$$\int \sin^2 x \cos^2 x \cos x\,dx$$

$$= \int \underline{\sin^2 x}\,(1-\underline{\sin^2 x})\,\underline{\cos x\,dx} \qquad \text{let } u = \underline{\sin x} \qquad du = \underline{\cos x\,dx}$$

$$= \int u^2(1-u^2)\,du$$

$$= \int u^2 - u^4\,du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C}$$

2. $\int sin^3(x)\ dx$

$$\int \sin^2 x \sin x\,dx$$

$$=(-1)\int (1-\underline{\cos^2 x})\,\underline{\overset{(-1)}{\sin x}\,dx} \qquad u = \underline{\cos x} \qquad du = \underline{-\sin x\,dx}$$

$$= -\int 1-u^2\,du$$

$$= -\left(u - \frac{u^3}{3}\right) + C$$

$$\boxed{\frac{\cos^3 x}{3} - \cos x + C}$$

3. $\int tan^6(x)sec^4(x)$

$$\int \tan^6 x \sec^2 x \sec^2 x \, dx$$

$$= \int \underline{\tan^6 x}\ \underline{\sec^2 x}\ (\underline{\tan^2 x}+1)\ \underline{dx} \qquad u = \underline{\tan x} \qquad du = \underline{\sec^2 x \, dx}$$

$$= \int u^6 (u^2+1) \, du$$

$$= \int u^8 + u^6 \, du$$

$$= \frac{u^9}{9} + \frac{u^7}{7} + C$$

$$= \boxed{\frac{\tan^9 x}{9} + \frac{\tan^7 x}{7} + C}$$

Solve the following problems using <u>trigonometric substitution:</u>

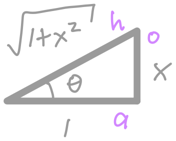
4. $\int \frac{x}{\sqrt{1+x^2}}\,dx$ $\quad x = 1\tan\theta \qquad dx = \sec^2\theta\, d\theta$

square root:

$$\sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$$

$$\int \frac{\tan\theta}{\sec\theta}\cdot \sec^2\theta\, d\theta = \int \tan\theta \sec\theta\, d\theta = \sec\theta + C$$

θ back to x: $x = \tan\theta \rightarrow \tan\theta = \frac{x}{1} \qquad \tan \frac{o}{a}$



$\sec = \frac{1}{\cos}$, $\cos \frac{a}{h}$ so $\sec \frac{h}{a}$

$$\sec\theta + C \rightarrow \frac{\sqrt{1+x^2}^{\,h}}{1_{\,a}} + C = \boxed{\sqrt{1+x^2} + C}$$

5. $\int_{0}^{3} \frac{x}{\sqrt{36-x^2}}\,dx$ $x = 6\sin\theta$ $dx = 6\cos\theta\,d\theta$

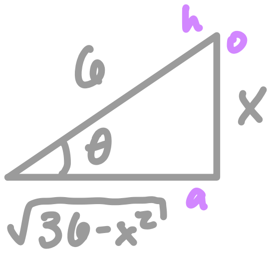
square root:

$$\sqrt{36-(6\sin\theta)^2} = \sqrt{36-36\sin^2\theta} = \sqrt{36(1-\sin^2\theta)}$$

$$= \sqrt{36\cos^2\theta} = 6\cos\theta$$

$$\int \frac{6\sin\theta}{\cancel{6\cos\theta}}\,\cancel{6\cos\theta}\,d\theta = \int 6\sin\theta\,d\theta = -6\cos\theta + C$$

Change θ back to x: $x = 6\sin\theta \rightarrow \sin\theta = \frac{x}{6}$ $\frac{o}{h}$



$\cos$ $\frac{a}{h}$

$$-6\cos\theta + C \rightarrow -\cancel{6}\left(\frac{\sqrt{36-x^2}}{\cancel{6}}\right) + C$$

$$\left[-\sqrt{36-x^2}\right]_{0}^{3} = -\sqrt{36-(3)^2} - \left(-\sqrt{36-0^2}\right)$$

plug in bounds!

$$= -\sqrt{27} + 6$$

$$= \boxed{6-\sqrt{27} \cong 0.804}$$

6. $\int \frac{\sqrt{4x^2-1}}{x}\,dx$

$$\int \frac{\sqrt{4(x^2-\frac{1}{4})}}{x}\,dx \qquad x=\frac{1}{2}\sec\theta \qquad dx=\frac{1}{4}\sec\theta\tan\theta\,d\theta$$

square root: $2\cdot\sqrt{x^2-\frac{1}{4}}$

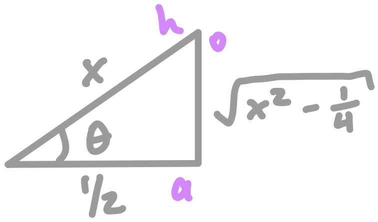
$$2\cdot\sqrt{(\tfrac{1}{2}\sec\theta)^2-\tfrac{1}{4}}=2\cdot\sqrt{\tfrac{1}{4}\sec^2\theta-\tfrac{1}{4}}=2\cdot\sqrt{\tfrac{1}{4}(\sec^2\theta-1)}$$

$$=2\cdot\sqrt{\tfrac{1}{4}\tan^2\theta}=2\cdot\tfrac{1}{2}\tan\theta=\tan\theta$$

$$\int\frac{\tan\theta}{\cancel{\frac{1}{2}\sec\theta}}\cdot\cancel{\tfrac{1}{2}\sec\theta}\tan\theta\,d\theta$$

$$=\int\tan^2\theta\,d\theta=\int\sec^2\theta-1\,d\theta=\underline{\tan\theta-\theta+C}$$

Change θ back to x: $x=\frac{1}{2}\sec\theta \rightarrow \sec\theta=\frac{x}{1/2}$ $\quad\frac{h}{a}$



$\tan\ \frac{o}{a}$

$\sec\theta=2x \rightarrow \theta=\sec^{-1}(2x)$

$$\tan\theta-\theta+C \rightarrow \frac{\sqrt{x^2-\frac{1}{4}}}{1/2}-\sec^{-1}(2x)+C=\boxed{2\sqrt{x^2-\tfrac{1}{4}}-\sec^{-1}(2x)+C}$$

Solve the following problems using <u>integration by partial fractions or polynomial division</u>:

7. $\int \frac{3x-2}{x+1}\,dx$

$$\begin{array}{r|l} & 3 - \frac{5}{x+1} \\ \hline x+1 & 3x-2 \\ & -(3x+3) \\ \cline{2-2} & \quad -5 \end{array}$$

$$\int 3 - \frac{5}{x+1}\,dx$$

$$= \boxed{3x - 5\ln|x-1| + C}$$

8. $\int \frac{5x+1}{2x^2-x-1}\,dx$

$$\frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

$$5x+1 = A(x-1) + B(2x+1)$$

$x=1$: $5(1)+1 = \cancel{A(1-1)} + B(2(1)+1)$

$$6 = 3B \rightarrow \underline{B=2}$$

$x=-\frac{1}{2}$: $5(-\frac{1}{2})+1 = A(-\frac{1}{2}-1) + \cancel{B(2(-\frac{1}{2})+1)}$

$$-\frac{3}{2} = -\frac{3}{2}A \rightarrow \underline{A=1}$$

plug in A & B

$$\int \frac{1}{2x+1} + \frac{2}{x-1}\,dx$$

$$\boxed{\frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C}$$

9. $\int \frac{x^2+1}{(x-3)(x-2)^2}\,dx$

$$\frac{x^2+1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$x^2+1 = A(x-2)^2 + B(x-3)(x-2) + C(x-3)$$

$x=2$: $(2)^2+1 = \cancel{A(2-2)^2} + \cancel{B(2-3)(2-2)} + C(2-3)$

$5 = -C \rightarrow \underline{C=-5}$

$x=3$: $(3)^2+1 = A(3-2)^2 + \cancel{B(3-3)(3-2)} + \cancel{C(3-3)}$

$\underline{10 = A}$

$$x^2+1 = 10(x-2)^2 + B(x-3)(x-2) - 5(x-3)$$

$x=0$: $1 = 10(-2)^2 + B(-3)(-2) - 5(-3)$

$1 = 40 + 6B + 15$

$-54 = 6B$

$\underline{B=-9}$

plug in A, B, and C

$$\int \frac{10}{x-3} + \frac{9}{x-2} + \underbrace{\frac{-5}{(x-2)^2}}\,dx$$

u-sub

let $u = x-2$ $du = dx$

$-5\int u^{-2}\,du = -5(-u^{-1})+C$

$= -5\left(-\frac{1}{x-2}\right)+C$

$$\boxed{10\ln|x-3| - 9\ln|x-2| + \frac{5}{x-2} + C}$$

$\Delta x = \frac{b-a}{n}$

$= \frac{4-1}{6} = \frac{1}{2}$

10. Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule to approximate $\int_1^4 \sqrt{ln(x)}dx$ with the value of n=6. (Round your answers to six decimal places.)

$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \dots 2f(x_{n-1}) + f(x_n)\right]$

$T_6 = \frac{1}{4}\left[f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + f(4)\right]$

$T_6 = \frac{1}{4}\left[\sqrt{\ln 1} + 2\sqrt{\ln 1.5} + 2\sqrt{\ln 2} + 2\sqrt{\ln 2.5} + 2\sqrt{\ln 3} + 2\sqrt{\ln 3.5} + \sqrt{\ln 4}\right]$

$\boxed{T_6 \cong 2.591334}$

$M_n = \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_n)\right]$

$M_6 = \frac{1}{2}\left[f(1.25) + f(1.75) + f(2.25) + f(2.75) + f(3.25) + f(3.75)\right]$

$M_6 = \frac{1}{2}\left[\sqrt{\ln 1.25} + \sqrt{\ln 1.75} + \sqrt{\ln 2.25} + \sqrt{\ln 2.75} + \sqrt{\ln 3.25} + \sqrt{\ln 3.75}\right]$

$\boxed{M_6 \cong 2.681046}$

$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4(x_1) + 2(x_2) + 4(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$

$S_6 = \frac{1}{6}\left[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)\right]$

$S_6 = \frac{1}{6}\left[\sqrt{\ln 1} + 4\sqrt{\ln 1.5} + 2\sqrt{\ln 2} + 4\sqrt{\ln 2.5} + 2\sqrt{\ln 3} + 4\sqrt{\ln 3.5} + \sqrt{\ln 4}\right]$

$\boxed{S_6 \cong 2.631976}$

11. What is the minimum value of n's it would take to estimate $\int_{0}^{4} x^{5} + 2x^{2}\,dx$ dx to an accuracy of 0.01 using the (a) Trapezoid Rule, (b) Midpoint Rule, and (c) Simpson's Rule?

$f(x) = x^5 + 2x^2$

$f'(x) = 5x^4 + 4x$

$f''(x) = 20x^3 + 4 \longrightarrow f''(4) = 1284$

$f'''(x) = 60x^2$

$f''''(x) = 120x \longrightarrow f''''(4) = 480$

$|E_T| \leq \frac{K(b-a)^3}{12n^2}$ (K = max value of $f''(x)$ within given bounds)

$0.01 > \frac{1280(4-0)^3}{12n^2} \rightarrow n^2 > \frac{82176}{0.12} \rightarrow n > 827.5$

*round up!

$\boxed{n = 828}$

$|E_M| \leq \frac{K(b-a)^3}{24n^2}$ (K = max value of $f''(x)$ within given bounds)

$0.01 > \frac{1280(4-0)^3}{24n^2} \rightarrow n^2 > \frac{82176}{0.24} \rightarrow n > 585.1$

*round up!

$\boxed{n = 586}$

$|E_S| \leq \frac{K(b-a)^5}{180n^4}$ (K = max value of $f''''(x)$ within given bounds)

$0.01 > \frac{480(4-0)^5}{180n^4} \rightarrow n^4 > \frac{491520}{1.8} \rightarrow n > 22.8$

*round up to nearest even number!

$\boxed{n = 24}$

12. Given $\int \frac{u^2}{\sqrt{a^2-\underline{u^2}}}\,du = -\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}(\frac{u}{a})+C$, find the integral of

$\int \frac{x^2}{\sqrt{5-\underline{4x^2}}}\,dx.$

set equal $u^2 = 4x^2 \rightarrow u = 2x$

$a^2 = 5 \rightarrow a = \sqrt{5}$

$u = 2x \rightarrow x = \frac{u}{2}$ $\qquad du = 2\,dx \rightarrow dx = \frac{du}{2}$

$u^2 = 4x^2$

plug in u + simplify

$$\int \frac{(\frac{u}{2})^2}{\sqrt{5-u^2}}\cdot\frac{du}{2}$$

$$= \int \frac{u^2}{4\sqrt{5-u^2}}\cdot\frac{du}{2}$$

$$= \frac{1}{8}\int \frac{u^2}{\sqrt{5-u^2}}\,du$$

$$= \frac{1}{8}\left(-\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}(\frac{u}{a})\right)+C$$

*plug in $\underline{u}$ + $\underline{a}$

$$= \boxed{\frac{1}{8}\left(-x\sqrt{5-4x^2}+\frac{5}{2}\sin^{-1}(\frac{2x}{\sqrt{5}})\right)+C}$$